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DYNAMIC DEFORMATION OF A THERMOVISCOELASTIC ROD OF TRIANGULAR CROSS SECTION IN A COUPLED FORMULATION

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For the coupled model of a thermoviscoelastic rod of equilateral triangular cross section, two exact solutions are obtained for the cases where a normal displacement and a shear stress or a tangential displacement and a normal stress are specified on the lateral surface of the rod. A dimensionless parameter R_0 is introduced to judge the appropriateness of taking into account the coupling in the formulation of the problem. Formulas are given for the velocities and lengths of the temperature, shear, and longitudinal waves, which can be used in experiments to determine the physical properties of thermoviscoelastic materials.

Key words: *dynamic deformation, coupled thermoviscoelasticity problems, rod.*

The properties of a thermoelastic body in a dynamic mode was the subject of research in [1–4] and other papers. The thermoviscoelastic model is a complex model, and dynamic problems have therefore been little studied. The exact solutions of dynamic problems for a two-dimensional thermoviscoelastic body are unknown.

1. Formulation of the Problem. unlike in the majority of linear models, in the thermoviscoelastic model, the mechanical properties of solids are most fully taken into account. Thermoviscoelastic properties are inherent in metals and their alloys under small variable mechanical and thermal loads [5]. Materials with such complex properties are described by various rheological models. For definiteness, we chose a model in which the elastic and viscous strain and strain rate tensors coincide and the total strains are the sum of the elastic and temperature strains. The stress tensor σ_{ij} is expressed in terms of the strain tensors e_{ij}, strain rates ε_{ij}, and temperature T as follows:

$$\sigma_{ij} = \lambda(e_{kk} - 3\alpha_t T)\delta_{ij} + 2\mu(e_{ij} - \alpha_t T\delta_{ij}) + \zeta(\varepsilon_{kk} - 3\alpha_t T_t)\delta_{ij} + 2\eta(\varepsilon_{ij} - \alpha_t T_t\delta_{ij}). \tag{1.1}$$

Here λ and μ are the Lamé elastic coefficients, ζ and η are the viscosity coefficients, α_t is the thermal-expansion coefficient, δ_{ij} is the unit Kronecker tensor, and $(\,\cdot\,)_t = \partial\,(\,\cdot\,)/\partial t$.

Below, we consider dynamic problems under plane strain conditions. Substituting σ_{ij} from (1.1) into the equations of motion for a continuum, we obtain the following two differential equations for the displacements u and v in Cartesian coordinates:

$$\lambda_0 u_{xx} + (\lambda+\mu)v_{xy} + \mu u_{yy} + \zeta_0 u_{txx} + (\zeta+\eta)v_{txy} + \eta u_{tyy} - \gamma_e T_x - \gamma_v T_{xt} = \rho u_{tt},$$

$$\lambda_0 = \lambda + 2\mu, \qquad \zeta_0 = \zeta + 2\eta, \tag{1.2}$$

$$\lambda_0 v_{yy} + (\lambda+\mu)u_{xy} + \mu v_{xx} + \zeta_0 v_{tyy} + (\zeta+\eta)u_{txy} + \eta v_{txx} - \gamma_e T_y - \gamma_v T_{yt} = \rho v_{tt},$$

$$\gamma_e = (3\lambda + 2\mu)\alpha_t, \qquad \gamma_v = (3\zeta + 2\eta)\alpha_t.$$

These equations should be supplemented by the heat-conduction equation

$$b\Delta T - k(u_{xt} + v_{yt}) = T_t, \qquad k = \gamma_e T_0/(C\rho). \tag{1.3}$$

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In (1.2) and (1.3), T_0 is the initial temperature, Δ is the Laplacian, ρ is the density, $(\,\cdot\,)_x = \partial\,(\,\cdot\,)/\partial x$, $(\,\cdot\,)_y = \partial\,(\,\cdot\,)/\partial y$, b is the thermal diffusivity, C is the specific heat, and k is the coupling coefficient (the term containing this quantity takes into account the temperature variation in the solid due to adiabatic volume variation [6]). For Eqs. (1.2) and (1.3), we specify two versions of conditions on the boundary Γ of the rod with an equilateral triangular cross section of height $2h$:

$$u_n\Big|_\Gamma = u_{10}\cos\omega t + u_{20}\sin\omega t, \qquad \tau_n\Big|_\Gamma = \tau_{10}\cos\omega t + \tau_{20}\sin\omega t,$$
$$\frac{\partial T}{\partial n}\Big|_\Gamma = q_{10}\cos\omega t + q_{20}\sin\omega t; \tag{1.4}$$

$$u_\tau\Big|_\Gamma = v_{10}\cos\omega t + v_{20}\sin\omega t, \qquad \sigma_n\Big|_\Gamma = \sigma_{10}\cos\omega t + \sigma_{20}\sin\omega t,$$
$$T\Big|_\Gamma = T_{10}\cos\omega t + T_{20}\sin\omega t. \tag{1.5}$$

Here u_n and u_τ are the normal displacement and the displacement tangential to the boundary Γ of the material points, τ_n and σ_n are the shear and normal stresses on the boundary of the rod, u_{j0}, τ_{j0}, v_{j0}, σ_{j0}, and T_{j0} $(j = 1, 2)$ are specified constants. Equations (1.1)–(1.5) constitute a linear problem. The material being deformed is heated with time due to energy dissipation, which can be taken into account by the nonlinear term $\sigma^v_{ij}\varepsilon^v_{ij}$ in the heat-conduction equation. For large values of t, the heating becomes substantial; therefore, the proposed linear model, which ignores dissipation, is suitable only for initial times.

We consider the problem of harmonic oscillations without initial conditions. The solution of this problem is sought in the form

$$u = U_1(x,y)\cos\omega t + U_2(x,y)\sin\omega t, \qquad v = V_1(x,y)\cos\omega t + V_2(x,y)\sin\omega t,$$
$$T = T_1(x,y)\cos\omega t + T_2(x,y)\sin\omega t, \tag{1.6}$$

where U_j, V_j, and T_j are the amplitudes of the displacement and temperature oscillations in the region Ω. Substitution of (1.6) into (1.2) and (1.3) yields the system

$$\lambda_0 U_{1xx} + (\lambda+\mu)V_{1xy} + \mu U_{1yy} + \omega\zeta_0 U_{2xx} + \omega(\zeta+\eta)V_{2xy} + \omega\eta U_{2yy} - \gamma_e T_{1x} - \omega\gamma_v T_{2x} + \rho\omega^2 U_1 = 0,$$
$$\lambda_0 U_{2xx} + (\lambda+\mu)V_{2xy} + \mu U_{2yy} - \omega\zeta_0 U_{1xx} - \omega(\zeta+\eta)V_{1xy} - \omega\eta U_{1yy} - \gamma_e T_{2x} + \omega\gamma_v T_{1x} + \rho\omega^2 U_2 = 0; \tag{1.7}$$

$$\lambda_0 V_{1yy} + (\lambda+\mu)U_{1xy} + \mu V_{1xx} + \omega\zeta_0 V_{2yy} + \omega(\zeta+\eta)U_{2xy} + \omega\eta V_{2xx} - \gamma_e T_{1y} - \omega\gamma_v T_{2y} + \rho\omega^2 V_1 = 0,$$
$$\lambda_0 V_{2yy} + (\lambda+\mu)U_{2xy} + \mu V_{2xx} - \omega\zeta_0 V_{1yy} - \omega(\zeta+\eta)U_{1xy} - \omega\eta V_{1xx} - \gamma_e T_{2y} + \omega\gamma_v T_{1y} + \rho\omega^2 V_2 = 0; \tag{1.8}$$

$$b\Delta T_1 - \omega k(U_{2x} + V_{2y}) - \omega T_2 = 0, \qquad b\Delta T_2 + \omega k(U_{1x} + V_{1y}) + \omega T_1 = 0. \tag{1.9}$$

2. Solution for a Flat Strip. In this case, we assume that the quantities U_j, V_j and T_j $(j = 1, 2)$ depend only on the coordinate x. We introduce the following notation:

$$U_j = P_j(x), \qquad V_j = Q_j(x), \qquad T_j = R_j(x) \qquad (j = 1, 2).$$

Equations (1.7)–(1.9) are simplified:

$$\lambda_0 P_1'' + \omega\zeta_0 P_2'' - \gamma_e R_1' - \omega\gamma_v R_2' + \rho\omega^2 P_1 = 0, \quad \lambda_0 P_2'' - \omega\zeta_0 P_1'' - \gamma_e R_2' + \omega\gamma_v R_1' + \rho\omega^2 P_2 = 0,$$
$$bR_1'' - \omega k P_2' - \omega R_2 = 0, \quad bR_2'' + \omega k P_1' + \omega R_1 = 0; \tag{2.1}$$

$$\mu Q_1'' + \omega\eta Q_2'' + \rho\omega^2 Q_1 = 0, \qquad \mu Q_2'' - \omega\eta Q_1'' + \rho\omega^2 Q_2 = 0. \tag{2.2}$$

Here the unknown functions P_j and R_j enter system (2.1) because of the coupling nature of the model, and for Q_j we have separate independent equations (2.2). Particular solutions of system (2.1), (2.2) are sought in the form

$$P_j = A_j\,\mathrm{e}^{\alpha x}, \qquad Q_j = B_j\,\mathrm{e}^{\beta x}, \qquad R_j = C_j\,\mathrm{e}^{\alpha x} \qquad (j = 1, 2). \tag{2.3}$$

Substitution of (2.3) into (2.1) and (2.2) yields the following system of equations for A_j, B_j, C_j, α, and β:

$$\lambda_0\alpha^2 A_1 + \omega\zeta_0\alpha^2 A_2 - \gamma_e\alpha C_1 - \omega\gamma_v\alpha C_2 + \rho\omega^2 A_1 = 0,$$
$$\lambda_0\alpha^2 A_2 - \omega\zeta_0\alpha^2 A_1 - \gamma_e\alpha C_2 + \omega\gamma_v\alpha C_1 + \rho\omega^2 A_2 = 0, \tag{2.4}$$
$$b\alpha^2 C_1 - \omega k\alpha A_2 - \omega C_2 = 0, \qquad b\alpha^2 C_2 + \omega k\alpha A_1 + \omega C_1 = 0;$$

$$\mu\beta^2 B_1 + \omega\eta\beta^2 B_2 + \rho\omega^2 B_1 = 0, \qquad \mu\beta^2 B_2 - \omega\eta\beta^2 B_1 + \rho\omega^2 B_2 = 0. \tag{2.5}$$

We first consider the simpler system (2.5). Equating its determinant to zero, we find four complex roots of the characteristic equation:

$$\beta_{1,2} = \pm(\alpha_{00} - i\beta_{00}), \qquad \beta_{3,4} = \pm(\alpha_{00} + i\beta_{00}), \qquad \beta_{1,2} = \bar{\beta}_{3,4},$$
$$\alpha_{00} = \omega\sqrt{\frac{\rho}{2\mu}}\,\frac{\sqrt{G_\eta - 1}}{G_\eta}, \qquad \beta_{00} = \omega\sqrt{\frac{\rho}{2\mu}}\,\frac{\sqrt{G_\eta + 1}}{G_\eta}, \qquad G_\eta = \sqrt{1 + \Big(\frac{\omega\eta}{\mu}\Big)^2}\,. \tag{2.6}$$

Here the bar above denotes coupling. To obtain the general solution of system (2.5) in explicit form, it is necessary to determine the coupling between the coefficients B_1 and B_2 for various values of $\beta = \beta_m$ $(m = 1, \ldots, 4)$. We introduce the following notation:

$$B_1(\beta_m) = B_{1m}, \qquad B_2(\beta_m) = B_{2m} \qquad (m = 1, \ldots, 4).$$

The coefficients B_{2m} can be expressed in terms of the coefficients B_{1m}, which will be treated as complex constants. Substituting $\beta = \beta_m$ $(m = 1, \ldots, 4)$ into (2.5), we find the desired couplings:

$$B_{2j} = -iB_{1j}, \qquad B_{2\,(j+2)} = iB_{1(j+2)}, \qquad j = 1, 2. \tag{2.7}$$

The general solution of system (2.2) becomes

$$Q_1(x) = \sum_{m=1}^{4} B_{1m}\,\mathrm{e}^{\beta_m x}, \qquad Q_2(x) = i\sum_{m=1}^{2}(B_{1(m+2)}\,\mathrm{e}^{\beta_{m+2}x} - B_{1m}\,\mathrm{e}^{\beta_m x}). \tag{2.8}$$

The right sides of equalities (2.8) contain complex quantities, whereas $Q_1(x)$ and $Q_2(x)$ are real functions of the real variable x. Therefore, these equations need to be reduced to a form that does not contain imaginary terms. For this, we associate each complex conjugate pair of characteristic roots β_m and $\beta_{m+2} = \bar{\beta}_m$ in (2.6) with a pair of complex conjugate coefficients:

$$B_{11} = \frac{D_1 + iD_2}{2}, \quad B_{13} = \frac{D_1 - iD_2}{2}, \quad B_{12} = \frac{D_3 - iD_4}{2}, \quad B_{14} = \frac{D_3 + iD_4}{2}. \tag{2.9}$$

Formulas (2.7) and (2.9) allow one to establish the following property: the sum of two terms in expressions (2.8) that correspond to two complex conjugate characteristic roots β_m and $\beta_{m+2} = \bar{\beta}_m$ $(m = 1, 2)$ is a real function. We show this using the expression for $Q_1(x)$ as an example:

$$B_{11}\,\mathrm{e}^{\beta_1 x} + B_{13}\,\mathrm{e}^{\beta_3 x} = (D_1 + iD_2)(\cos\beta_{00}x - i\sin\beta_{00}x)\,\mathrm{e}^{\alpha_{00}x}/2$$
$$+ (D_1 - iD_2)(\cos\beta_{00}x + i\sin\beta_{00}x)\,\mathrm{e}^{\alpha_{00}x}/2 = (D_1\cos\beta_{00}x + D_2\sin\beta_{00}x)\,\mathrm{e}^{\alpha_{00}x}. \tag{2.10}$$

In view of the property (2.10), the general solution (2.8) for a flat strip can be written in real form. If the variable x is replaced by the difference $x - h$ (which is reasonable for subsequent calculations), the expressions for $Q_1(x)$ and $Q_2(x)$ become

$$Q_1(x) = [D_1\cos\beta_{00}(x-h) + D_2\sin\beta_{00}(x-h)]\,\mathrm{e}^{\alpha_{00}(x-h)} + [D_3\cos\beta_{00}(x-h) + D_4\sin\beta_{00}(x-h)]\,\mathrm{e}^{\alpha_{00}(h-x)},$$
$$Q_2(x) = [D_2\cos\beta_{00}(x-h) - D_1\sin\beta_{00}(x-h)]\,\mathrm{e}^{\alpha_{00}(x-h)} - [D_4\cos\beta_{00}(x-h) - D_3\sin\beta_{00}(x-h)]\,\mathrm{e}^{\alpha_{00}(h-x)}. \tag{2.11}$$

We now proceed to the solution of system (2.4). In finding the characteristic roots in explicit form from the determinant of this system, we obtain an algebraic equation of the eighth order, which should be written in compact form. For this, we find A_1 and A_2 from the second and third equalities of system (2.4):

$$\omega k\alpha A_1 = -\omega C_1 - b\alpha^2 C_2, \qquad \omega k\alpha A_2 = b\alpha^2 C_1 - \omega C_2. \tag{2.12}$$

Eliminating A_1 and A_2 from system (2.4) by using (2.12), we obtain two equations of the fourth order for α:

$$b\lambda_0\alpha^4 + (b\rho + \zeta_0 + k\gamma_v)\omega^2\alpha^2 = \pm i\omega[-\zeta_0 b\alpha^4 + (\lambda_0 + k\gamma_e)\alpha^2 + \rho\omega^2]. \tag{2.13}$$

From this, we find eight characteristic roots. We introduce the dimensionless parameters

$$N_0 = \frac{b\rho\omega}{\lambda_0}, \qquad M_e = \frac{k\gamma_e}{\lambda_0} = \frac{(3\lambda + 2\mu)^2\alpha_t^2 T_0}{C\rho(\lambda + 2\mu)}, \qquad M_v = \frac{k\gamma_v}{b\rho}, \qquad M_\zeta = \frac{\zeta_0}{b\rho}$$

and notation

$$A_* = N_0^2(1 + M_\zeta + M_v)^2 - (1 + M_e)^2 - 4N_0^2 M_\zeta, \quad B_* = 2N_0[1 - M_e - (1 + M_e)(M_\zeta + M_v)],$$

$$K_0 = \frac{1}{\sqrt{2}}\sqrt{\sqrt{A_*^2 + B_*^2} + A_*}\,, \qquad L_0 = \frac{1}{\sqrt{2}}\sqrt{\sqrt{A_*^2 + B_*^2} - A_*}\,.$$

If a given material with weak coupling obeys the inequality

$$M_e + (1 + M_e)(M_\zeta + M_v) = R_0 < 1, \tag{2.14}$$

the roots of Eq. (2.13) with the plus sign in can be written as

$$\alpha_k^2 = \frac{i(1 + M_e) - N_0(1 + M_\zeta + M_v) \pm (K_0 + iL_0)}{1 + iN_0M_\zeta}\,\frac{\omega}{2b}, \qquad k = 1, \dots, 4, \tag{2.15}$$

and the roots of Eq. (2.13) with the minus sign can be written as

$$\alpha_k^2 = \frac{-i(1 + M_e) - N_0(1 + M_\zeta + M_v) \pm (K_0 - iL_0)}{1 - iN_0M_\zeta}\,\frac{\omega}{2b}, \qquad k = 5, \dots, 8. \tag{2.16}$$

In the case $R_0 > 1$, where the coupling is substantial, the roots of Eq. (2.13) with the plus sign are written as

$$\alpha_k^2 = \frac{i(1 + M_e) - N_0(1 + M_\zeta + M_v) \pm (K_0 - iL_0)}{1 + iN_0M_\zeta}\,\frac{\omega}{2b}, \qquad k = 1, \dots, 4, \tag{2.17}$$

and the roots of the Eq. (2.13) with the minus sign are written as

$$\alpha_k^2 = \frac{-i(1 + M_e) - N_0(1 + M_\zeta + M_v) \pm (K_0 + iL_0)}{1 - iN_0M_\zeta}\,\frac{\omega}{2b}, \qquad k = 5, \dots, 8. \tag{2.18}$$

Next, it is expedient to represent the roots $\alpha_1, \dots, \alpha_8$ as follows:

$$\alpha_{1,2} = \pm(\alpha_{01} + i\beta_{01}), \qquad \alpha_{3,4} = \pm(\alpha_{03} + i\beta_{03}), \qquad \alpha_{5,6} = \bar{\alpha}_{1,2}, \qquad \alpha_{7,8} = \bar{\alpha}_{3,4}. \tag{2.19}$$

The real and imaginary parts of the roots are found from (2.15)–(2.18) using the formulas

$$R_0 < 1: \qquad \alpha_{01} = R_1^* \cos\varphi_1, \qquad \beta_{01} = R_1^* \sin\varphi_1,$$

$$\varphi_1 = \frac{1}{2}\,\arctan\frac{B_1^*}{A_1^*}, \qquad R_1^* = \sqrt{\omega\sqrt{A_1^{*2} + B_1^{*2}}\Big/[2b(1 + N_0^2M_\zeta^2)]}\,,$$

$$A_1^* = K_0 - N_0(1 + M_\zeta + M_v) + N_0M_\zeta(1 + M_e + L_0),$$

$$B_1^* = 1 + M_e + L_0 + N_0M_\zeta[N_0(1 + M_\zeta + M_v) - K_0],$$

$$R_0 < 1: \quad \alpha_{03} = R_3^* \cos\varphi_3, \qquad \beta_{03} = R_3^* \sin\varphi_3,$$

$$\varphi_3 = \frac{1}{2}\,\arctan\frac{B_3^*}{A_3^*}, \qquad R_3^* = \sqrt{\omega\sqrt{A_3^{*2} + B_3^{*2}}\Big/[2b(1 + N_0^2M_\zeta^2)]}\,,$$

$$A_3^* = -K_0 - N_0(1 + M_\zeta + M_v) + N_0M_\zeta(1 + M_e - L_0),$$

$$B_3^* = 1 + M_e - L_0 + N_0M_\zeta[N_0(1 + M_\zeta + M_v) - K_0],$$

$$R_0 > 1: \quad \alpha_{01} = R_1^* \cos\varphi_1, \qquad \beta_{01} = R_1^* \sin\varphi_1,$$

$$\varphi_1 = \frac{1}{2}\,\arctan\,\frac{B_1^*}{A_1^*}, \qquad R_1^* = \sqrt{\omega\sqrt{A_1^{*2}+B_1^{*2}}\Big/\,[2b(1+N_0^2M_\zeta^2)]}\,,$$

$$A_1^* = K_0 - N_0(1+M_\zeta+M_v) + N_0M_\zeta(1+M_e-L_0),$$

$$B_1^* = 1+M_e-L_0+N_0M_\zeta[N_0(1+M_\zeta+M_v)-K_0],$$

$$R_0>1:\quad \alpha_{03} = R_3^*\cos\varphi_3, \qquad \beta_{03} = R_3^*\sin\varphi_3,$$

$$\varphi_3 = \frac{1}{2}\,\arctan\,\frac{B_3^*}{A_3^*}, \qquad R_3^* = \sqrt{\omega\sqrt{A_3^{*2}+B_3^{*2}}\Big/\,[2b(1+N_0^2M_\zeta^2)]}\,,$$

$$A_3^* = -K_0 - N_0(1+M_\zeta+M_v) + N_0M_\zeta(1+M_e+L_0),$$

$$B_3^* = 1+M_e+L_0+N_0M_\zeta[N_0(1+M_\zeta+M_v)-K_0].$$

To obtain the general solution of system (2.1) in explicit form, it is necessary to refine the couplings between the coefficients A_j and C_j $(j=1,2)$ for various values of $\alpha=\alpha_m$ $(m=1,\ldots,8)$. For this purpose, we introduce the following notation:

$$A_j = A_j(\alpha_m), \quad C_j = C_j(\alpha_m) \quad (j=1,2,\ m=1,\ldots,8), \qquad C_1(\alpha_m) = H_m.$$

We express the coefficients $C_2(\alpha_m)$ and $A_j(\alpha_m)$ in terms of the quantities H_m, which will be considered complex. Substitution of $\alpha=\alpha_m$ into (2.4) and (2.12) yields

$$C_2(\alpha_m) = iC_1(\alpha_m) = iH_m, \qquad C_2(\alpha_{m+4}) = -iC_1(\alpha_{m+4}) = -iH_{m+4},$$

$$A_1(\alpha_m) = -\Big(i\,\frac{b\alpha_m}{\omega} + \frac{\bar\alpha_m}{|\alpha_m|^2}\Big)\frac{H_m}{k}, \qquad A_1(\alpha_{m+4}) = \Big(i\,\frac{b\alpha_{m+4}}{\omega} - \frac{\bar\alpha_{m+4}}{|\alpha_{m+4}|^2}\Big)\frac{H_{m+4}}{k},$$
$$A_2(\alpha_m) = iA_1(\alpha_m), \qquad A_2(\alpha_{m+4}) = -iA_1(\alpha_{m+4}) \qquad (m=1,\ldots,4).$$

As a result, the general solution of system (2.1) becomes

$$P_1(x) = -\sum_{m=1}^{4}\Big(i\,\frac{b\alpha_m}{\omega} + \frac{\bar\alpha_m}{|\alpha_m|^2}\Big)\frac{H_m}{k}\,\mathrm{e}^{\alpha_m x} + \sum_{m=5}^{8}\Big(i\,\frac{b\alpha_m}{\omega} - \frac{\bar\alpha_m}{|\alpha_m|^2}\Big)\frac{H_m}{k}\,\mathrm{e}^{\alpha_m x},$$
$$P_2(x) = \sum_{m=1}^{4}\Big(\frac{b\alpha_m}{\omega} - i\,\frac{\bar\alpha_m}{|\alpha_m|^2}\Big)\frac{H_m}{k}\,\mathrm{e}^{\alpha_m x} + \sum_{m=5}^{8}\Big(\frac{b\alpha_m}{\omega} + i\,\frac{\bar\alpha_m}{|\alpha_m|^2}\Big)\frac{H_m}{k}\,\mathrm{e}^{\alpha_m x}, \tag{2.20}$$
$$R_1(x) = \sum_{m=1}^{8} H_m\,\mathrm{e}^{\alpha_m x}, \qquad R_2(x) = i\sum_{m=1}^{4} H_m\,\mathrm{e}^{\alpha_m x} - i\sum_{m=5}^{8} H_m\,\mathrm{e}^{\alpha_m x}.$$

For the four pairs of complex-conjugate characteristic roots (2.19), we introduce the corresponding pairs of complex-conjugate coefficients:

$$H_m = (A_{0m} - iC_{0m})/2, \qquad H_{m+4} = \bar H_m = (A_{0m} + iC_{0m})/2, \qquad m=1,\ldots,4.$$

Here A_{0m} and C_{0m} $(m=1,\ldots,4)$ are eight unknowns, which are then found from boundary conditions (1.4) or (1.5). In (2.10), it is shown that the sum of two terms in expressions (2.20) that correspond to the two complex-conjugate characteristic roots α_m and α_{m+4} $(m=1,\ldots,4)$, is a real function. For a more compact form of the subsequent expressions, we introduce the auxiliary constants p_j and q_j and the notation of the real and imaginary parts of the characteristic roots with even subscripts:

$$p_j = \frac{1}{k}\Big(\frac{\beta_{0j}}{R_j^{*2}} - \frac{b\alpha_{0j}}{\omega}\Big), \qquad q_j = \frac{1}{k}\Big(\frac{\alpha_{0j}}{R_j^{*2}} - \frac{b\beta_{0j}}{\omega}\Big), \qquad j=1,3,$$
$$p_{2k} = p_{2k-1}, \qquad q_{2k} = q_{2k-1}, \qquad \alpha_{0(2k)} = \alpha_{0(2k-1)}, \qquad \beta_{0(2k)} = \beta_{0(2k-1)}, \qquad k=1,2. \tag{2.21}$$

With the use of the property (2.10) and the notation (2.21), the general solution in (2.20) for a flat strip is reduced to real form. If the variable x in (2.20) is replaced by the difference $x-h$ (which proves more convenient for making the solution to satisfy the boundary conditions), the expressions for $P_j(x)$ and $R_j(x)$ become

$$R_1(x) = \sum_{k=1}^{4} [A_{0k} \cos\beta_{0k}(x-h) - (-1)^k C_{0k} \sin\beta_{0k}(x-h)]\, \mathrm{e}^{(-1)^k \alpha_{0k}(h-x)},$$

$$R_2(x) = \sum_{k=1}^{4} [C_{0k} \cos\beta_{0k}(x-h) + (-1)^k A_{0k} \sin\beta_{0k}(x-h)]\, \mathrm{e}^{(-1)^k \alpha_{0k}(h-x)},$$

$$P_1(x) = \sum_{k=1}^{4} \{q_k[(-1)^k A_{0k} \cos\beta_{0k}(x-h) - C_{0k} \sin\beta_{0k}(x-h)] \tag{2.22}$$

$$- p_k[(-1)^k C_{0k} \cos\beta_{0k}(x-h) + A_{0k} \sin\beta_{0k}(x-h)]\}\, \mathrm{e}^{(-1)^k \alpha_{0k}(h-x)},$$

$$P_2(x) = \sum_{k=1}^{4} \{p_k[(-1)^k A_{0k} \cos\beta_{0k}(x-h) - C_{0k} \sin\beta_{0k}(x-h)]$$

$$+ q_k[(-1)^k C_{0k} \cos\beta_{0k}(x-h) + A_{0k} \sin\beta_{0k}(x-h)]\}\, \mathrm{e}^{(-1)^k \alpha_{0k}(h-x)}.$$

The general integrals for a thermoviscoelastic strip (2.11), (2.22) contain 12 arbitrary constants A_{0j}, C_{0j}, and D_j ($j = 1, \ldots, 4$) which are found from the conditions on the boundaries of the flat strip. The functions obtained will be used to construct two exact solutions for a rod of triangular cross section.

3. First Exact Solution. To construct the solution, we use a special procedure based on the variables ξ [7], which are determined as follows. We denote the radius-vectors of a certain pole and an arbitrary point in the section of the rod Ω by $\boldsymbol{r}_0$ and $\boldsymbol{r}$ and the radius-vectors of the vertices of the equilateral triangle Ω of height $2h$ by $\boldsymbol{r}_m$ and introduce the auxiliary variables ξ and ξ_k:

$$\xi = (\boldsymbol{r} - \boldsymbol{r}_0)\boldsymbol{n}, \qquad \xi_m = (\boldsymbol{r} - \boldsymbol{r}_m)\boldsymbol{n}_m, \qquad m = 1, 2, 3 \tag{3.1}$$

($\boldsymbol{n}$ is a certain unit vector, $\boldsymbol{n}_k$ are the inward unit normals to the sides of the triangle Ω, whose vertices and sides are numbered counter-clockwise). With this definition of the variables ξ_m, the equations of the sides of the triangle are given by the equalities $\xi_1 = 0$, $\xi_2 = 0$, and $\xi_3 = 0$. For the points $(x, y) \in \Omega$, the strict inequalities $\xi_1 > 0$, $\xi_2 > 0$, and $\xi_3 > 0$ hold. The variables ξ and ξ_m and the normals $\boldsymbol{n}_m$ on the plane (x, y) possess the following properties, which will be used in the subsequent analysis:

$$\begin{gathered} \boldsymbol{n}_1 + \boldsymbol{n}_2 + \boldsymbol{n}_3 = 0, \qquad \boldsymbol{n}_1\boldsymbol{n}_2 = \boldsymbol{n}_1\boldsymbol{n}_3 = \boldsymbol{n}_2\boldsymbol{n}_3 = -1/2, \\ \boldsymbol{n}_1 \times \boldsymbol{n}_2 = \boldsymbol{n}_2 \times \boldsymbol{n}_3 = \boldsymbol{n}_3 \times \boldsymbol{n}_1 = \sqrt{3}/2, \qquad \xi_1 + \xi_2 + \xi_3 = 2h; \end{gathered} \tag{3.2}$$

$$\begin{gathered} F = F(\xi) \in C^2(\Omega): \qquad F_x = F'(\xi) n_x, \qquad F_y = F'(\xi) n_y, \\ F_{xx} = F''(\xi) n_x^2, \qquad F_{xy} = F''(\xi) n_x n_y, \qquad F_{yy} = F''(\xi) n_y^2. \end{gathered} \tag{3.3}$$

Here $\boldsymbol{n}_j \times \boldsymbol{n}_k$ is the unique nonzero projection of the vector product onto the z axis. Using the functions $R_j(\xi)$, $P_j(\xi)$, and $Q_j(\xi)$ obtained by formulas (2.11) and (2.22), it is possible to construct a particular solution of system (1.7)–(1.9):

$$\begin{gathered} U_j(x, y) = P_j(\xi) n_x - Q_j(\xi) n_y, \qquad V_j(x, y) = P_j(\xi) n_y + Q_j(\xi) n_x, \\ T_j(x, y) = R_j(\xi), \qquad j = 1, 2. \end{gathered} \tag{3.4}$$

The forms of the functions U_j, V_j, and T_j in (3.4) differ significantly. This is explained by the fact that (U_j, V_j) is a vector function and T_j is a scalar function. Transformation from x to the variable ξ is equivalent to rotation of the coordinate system. In this case, vector functions are transformed under the laws of vector algebra and scalar functions do not change; therefore, the functions (U_j, V_j) contain projections of the normal vector n_x

and n_y that take into account the rotation, and the functions T_j do not contain these projections in similar form. Below, the following properties will be used.

Property 1. If the functions $P_j(x)$, $Q_j(x)$, and $R_j(x)$ used in expressions (3.4) are solutions of systems (2.1) and (2.2), i.e., if they have the form (2.11) and (2.22), then U_j, V_j, and T_j from (3.4) satisfy all differential equations of system (1.7)–(1.9).

Property 2. In (3.4), the functions $Q_j(\xi)$ as particular solutions of Eqs. (2.2) can be chosen independently of the particular solutions $P_j(\xi)$ and $R_j(\xi)$.

To prove Properties 1 and 2, we substitute U_j, V_j, and T_j from (3.4) into the first equations of (1.7) and (1.9); for the remaining equations, similar manipulations can be performed. Using the expressions for the particular derivatives from (3.3), we obtain

$$\lambda_0(P_1''n_x^3 - Q_1''n_x^2n_y) + (\lambda+\mu)(P_1''n_xn_y^2 + Q_1''n_x^2n_y) + \mu(P_1''n_xn_y^2 - Q_1''n_y^3)$$
$$+ \omega\zeta_0(P_2''n_x^3 - Q_2''n_x^2n_y) + \omega(\zeta+\eta)(P_2''n_xn_y^2 + Q_2''n_x^2n_y)$$
$$+ \omega\eta(P_2''n_xn_y^2 - Q_2''n_y^3) - \gamma_eR_1'n_x - \omega\gamma_vR_2'n_x + \rho\omega^2(P_1n_x - Q_1n_y) = 0, \tag{3.5}$$
$$bR_1'' - k\omega(P_2'n_x^2 - Q_2'n_xn_y) - k\omega(P_2'n_y^2 + Q_2'n_xn_y) - \omega R_2 = 0.$$

After simplifications, the last equation in (3.5) coincides with the third equation in (2.1.) In the first equation of (3.5), we group all terms ahead of P_j and Q_j:

$$P_1''\,n_x(\lambda_0n_x^2 + (\lambda+\mu)n_y^2 + \mu n_y^2) + \omega P_2''n_x(\zeta_0n_x^2 + (\zeta+\eta)n_y^2 + \eta n_y^2)$$
$$+ \rho\omega^2n_xP_1 - Q_1''\,n_y(\lambda_0n_x^2 - (\lambda+\mu)n_x^2 + \mu n_y^2) - \gamma_eR_1'n_x - \omega\gamma_vR_2'n_x$$
$$- \omega Q_2''n_y(\zeta_0n_x^2 - (\zeta+\eta)n_x^2 + \eta n_y^2) - \rho\omega^2n_yQ_1 = 0. \tag{3.6}$$

The coefficients at P_j'' and Q_j'' are transformed by the formulas

$$\lambda_0n_x^2 + (\lambda+\mu)n_y^2 + \mu n_y^2 = \lambda_0n_x^2 + \lambda_0n_y^2 = \lambda_0,$$
$$\lambda_0n_x^2 - (\lambda+\mu)n_x^2 + \mu n_y^2 = \mu n_x^2 + \mu n_y^2 = \mu. \tag{3.7}$$

With the use of (3.7), Eq. (3.6) is reduced to the form

$$n_x(\lambda_0P_1'' + \omega\zeta_0P_2'' - \gamma_eR_1' - \omega\gamma_vR_2' + \rho\omega^2P_1) - n_y(\mu Q_1'' + \omega\eta Q_2'' + \rho\omega^2Q_1) = 0. \tag{3.8}$$

Since P_j, Q_j, and R_j satisfy Eqs. (2.1) and (2.2) by construction, the expressions in parentheses in (3.8) vanish. Thus, Properties 1 and 2 are proved. If on the right sides of expressions (3.4), the variable ξ is replaced by any of the variables ξ_m determined in (3.1) the expressions obtained for U_j, V_j, and T_j satisfy system (1.7)–(1.9).

In writing the exact solution, we introduce the functions

$$P_j^{(a)}(\xi) = P_j(\xi) - P_j(2h-\xi), \qquad R_j^{(s)}(\xi) = R_j(\xi) + R_j(2h-\xi), \qquad j = 1, 2.$$

The functions $P_j^{(s)}(\xi)$, $R_j^{(a)}(\xi)$, $Q_j^{(s)}(\xi)$, and $Q_j^{(a)}(\xi)$ are introduced similarly. The superscript (s) [or (a)] indicates that the function is symmetric (or antisymmetric) about the point $\xi = h$; therefore, for these functions and their derivatives, the following equalities are satisfied:

$$P_j^{(a)}(\xi) + P_j^{(a)}(2h-\xi) = 0, \qquad R_j^{(s)}(\xi) - R_j^{(s)}(2h-\xi) = 0, \qquad j = 1, 2,$$
$$P_j^{(a)\prime}(\xi) - P_j^{(a)\prime}(2h-\xi) = 0, \qquad R_j^{(s)\prime}(\xi) + R_j^{(s)\prime}(2h-\xi) = 0. \tag{3.9}$$

If the functions $P_j(\xi)$ and $R_j(\xi)$ jointly contain eight constants, the functions $P_j^{(s)}(\xi)$ and $R_j^{(a)}(\xi)$ contain only four constants and the functions $Q_j^{(a)}(\xi)$ contain two constants; we denote these constants by $F_1, \ldots, F_4$ and G_1 and G_2:

$$F_1 = 2(A_{01} + A_{02}), \quad F_2 = 2(C_{01} + C_{02}), \quad F_3 = 2(A_{03} + A_{04}), \quad F_4 = 2(C_{03} + C_{04}),$$
$$G_1 = 2(D_1 + D_3), \qquad G_2 = 2(D_2 - D_4).$$

Below, a particular form of the functions $P_j^{(a)}(\xi)$, $Q_j^{(s)}(\xi)$, and $R_j^{(s)}(\xi)$ will be required. To obtain a compact form of these functions, we introduce the notation

$$\mathrm{coSh}_{0j}(\xi) = \cos\beta_{0j}(\xi - h)\sinh\,\alpha_{0j}(\xi - h), \qquad \mathrm{siCh}_{0j}(\xi) = \sin\beta_{0j}(\xi - h)\cosh\,\alpha_{0j}(\xi - h),$$

$$\mathrm{coCh}_{0j}(\xi) = \cos\beta_{0j}(\xi - h)\cosh\,\alpha_{0j}(\xi - h), \qquad \mathrm{siSh}_{0j}(\xi) = \sin\beta_{0j}(\xi - h)\sinh\,\alpha_{0j}(\xi - h).$$

In these notation, the functions $P_j^{(a)}(\xi)$, $Q_j^{(s)}(\xi)$, and $R_j^{(s)}(\xi)$ are written as

$$P_1^{(a)}(\xi) = \sum_{k=1}^{2}\{p_{2k-1}[F_{2k}\,\mathrm{coSh}_{0(2k-1)}(\xi) - F_{2k-1}\,\mathrm{siCh}_{0(2k-1)}(\xi)]$$
$$- q_{2k-1}[F_{2k-1}\,\mathrm{coSh}_{0(2k-1)}(\xi) + F_{2k}\,\mathrm{siCh}_{0(2k-1)}(\xi)]\},$$

$$P_2^{(a)}(\xi) = -\sum_{k=1}^{2}\{p_{2k-1}[F_{2k-1}\,\mathrm{coSh}_{0(2k-1)}(\xi) + F_{2k}\,\mathrm{siCh}_{0(2k-1)}(\xi)]$$
$$+ q_{2k-1}[F_{2k-1}\,\mathrm{siCh}_{0(2k-1)}(\xi) - F_{2k}\,\mathrm{coSh}_{0(2k-1)}(\xi)]\};$$

$$Q_1^{(s)}(\xi) = G_1\,\mathrm{coCh}_{00}(\xi) + G_2\,\mathrm{siSh}_{00}(\xi), \qquad Q_2^{(s)}(\xi) = G_2\,\mathrm{coCh}_{00}(\xi) - G_1\,\mathrm{siSh}_{00}(\xi); \tag{3.10}$$

$$R_1^{(s)}(\xi) = F_1\,\mathrm{coCh}_{01}(\xi) + F_2\,\mathrm{siSh}_{01}(\xi) + F_3\,\mathrm{coCh}_{03}(\xi) + F_4\,\mathrm{siSh}_{03}(\xi),$$

$$R_2^{(s)}(\xi) = F_2\,\mathrm{coCh}_{01}(\xi) - F_1\,\mathrm{siSh}_{01}(\xi) + F_4\,\mathrm{coCh}_{03}(\xi) - F_3\,\mathrm{siSh}_{03}(\xi).$$

The solution of problem (1.7)–(1.9) with boundary conditions (1.4) is represented as the sums

$$U_j(x,y) = \sum_{k=1}^{3}[P_j^{(a)}(\xi_k)n_{kx} - Q_j^{(s)}(\xi_k)n_{ky}], \qquad T_j(x,y) = \sum_{k=1}^{3} R_j^{(s)}(\xi_k),$$

$$V_j(x,y) = \sum_{k=1}^{3}[P_j^{(a)}(\xi_k)n_{ky} + Q_j^{(s)}(\xi_k)n_{kx}], \qquad j = 1,2. \tag{3.11}$$

By virtue of Properties 1 and 2, the functions U_j, V_j, and T_j in (3.11) satisfy Eqs. (1.7)–(1.9). It remains to satisfy boundary conditions (1.4), which previously need to be transformed. For this, we write the normal displacement component $u_n|_\Gamma = (un_x + vn_y)|_\Gamma$ on the boundary Γ in the form

$$(U_j n_x + V_j n_y)\Big|_\Gamma = u_{j0}, \qquad j = 1,2. \tag{3.12}$$

In these problems, it is assumed that all analytical relations similar to (3.11) are equally valid for the sides of the equilateral triangle; therefore, it is sufficient that all boundary conditions are satisfied on one side, for example, on the side $\xi_3 = 0$. Then, on the other two sides of the triangle for $\xi_1 = 0$ or $\xi_2 = 0$, the boundary conditions are satisfied automatically. For the points (x,y) on the side of the triangle $\xi_3 = 0$ between the variables ξ_1 and ξ_2, we have

$$\xi_3 = 0\colon \qquad \xi_1 + \xi_2 = 2h. \tag{3.13}$$

Substituting U_j and V_j from (3.11) into (3.12) for $\xi_3 = 0$ and using (3.13), we obtain the following two equations:

$$[P_j^{(a)}(\xi_1)(\boldsymbol{n}_1\boldsymbol{n}_3) + P_j^{(a)}(2h - \xi_1)(\boldsymbol{n}_2\boldsymbol{n}_3)] + P_j^{(a)}(0)$$
$$+ [Q_j^{(s)}(\xi_1)\boldsymbol{n}_1 \times \boldsymbol{n}_3 + Q_j^{(s)}(2h - \xi_1)\boldsymbol{n}_2 \times \boldsymbol{n}_3] = u_{j0} \qquad (j = 1,2).$$

Using properties (3.2) and (3.9), it is easy show that all terms containing the variable ξ_1 in square brackets are mutually cancelled; therefore,

$$P_j^{(a)}(0) = u_{j0} \qquad (j = 1,2). \tag{3.14}$$

We consider the boundary condition in (1.4) for the heat flux on the side $\xi_3 = 0$. After substitution of $T_j(x,y)$ from (3.11), this condition becomes

$$[R_j^{(s)\prime}(\xi_1)(\boldsymbol{n}_1\boldsymbol{n}_3) + R_j^{(s)\prime}(2h-\xi_1)(\boldsymbol{n}_2\boldsymbol{n}_3)] + R_j^{(s)\prime}(0) = q_{j0}, \qquad j = 1,2. \tag{3.15}$$

Using (3.2) and (3.9), one can prove that the expression in square brackets in (3.15) vanishes and, hence,

$$R_j^{(s)\prime}(0) = q_{j0}, \qquad j = 1,2. \tag{3.16}$$

The system of four equations (3.14), (3.16) for the coefficients $F_1,\ldots,F_4$ is represented in explicit form

$$\begin{gathered}
\sum_{k=1}^{2}\{p_{2k-1}[F_{2k-1}\operatorname{siCh}_{0(2k-1)}(h) - F_{2k}\operatorname{coSh}_{0(2k-1)}(h)] \\
+ q_{2k-1}[F_{2k-1}\operatorname{coSh}_{0(2k-1)}(h) + F_{2k}\operatorname{siCh}_{0(2k-1)}(h)]\} = u_{10}, \\
\sum_{k=1}^{2}\{p_{2k-1}[F_{2k-1}\operatorname{coSh}_{0(2k-1)}(h) + F_{2k}\operatorname{siCh}_{0(2k-1)}(h)] \\
+ q_{2k-1}[F_{2k}\operatorname{coSh}_{0(2k-1)}(h) - F_{2k-1}\operatorname{siCh}_{0(2k-1)}(h)]\} = u_{20}, \\
\sum_{k=1}^{2}\{F_{2k-1}[\beta_{0(2k-1)}\operatorname{siCh}_{0(2k-1)}(h) - \alpha_{0(2k-1)}\operatorname{coSh}_{0(2k-1)}(h)] \\
- F_{2k}[\beta_{0(2k-1)}\operatorname{coSh}_{0(2k-1)}(h) + \alpha_{0(2k-1)}\operatorname{siCh}_{0(2k-1)}(h)]\} = q_{10}, \\
\sum_{k=1}^{2}\{F_{2k-1}[\beta_{0(2k-1)}\operatorname{coSh}_{0(2k-1)}(h) + \alpha_{0(2k-1)}\operatorname{siCh}_{0(2k-1)}(h)] \\
+ F_{2k}[\beta_{0(2k-1)}\operatorname{siCh}_{0(2k-1)}(h) - \alpha_{0(2k-1)}\operatorname{coSh}_{0(2k-1)}(h)]\} = q_{20}.
\end{gathered} \tag{3.17}$$

The linear system (3.17) is easily solved on a computer. It remains to elucidate whether there are cases where the above equations have no solution. We show that the determinant of the system is always $\Delta_1^* > 0$. Using the properties of Eqs. (3.17), instead of $F_1^*,\ldots,F_4^*$ we introduce the new unknown complexes $x_1,\ldots,x_4$:

$$\begin{gathered}
x_1 = F_1\operatorname{siCh}_{01}(h) - F_2\operatorname{coSh}_{01}(h), \qquad x_2 = F_1\operatorname{coSh}_{01}(h) + F_2\operatorname{siCh}_{01}(h), \\
x_3 = F_3\operatorname{siCh}_{03}(h) - F_4\operatorname{coSh}_{03}(h), \qquad x_4 = F_3\operatorname{coSh}_{03}(h) + F_4\operatorname{siCh}_{03}(h).
\end{gathered} \tag{3.18}$$

In the notation (3.18), system (3.17) becomes simpler:

$$\begin{gathered}
\beta_{01}x_1 - \alpha_{01}x_2 + \beta_{03}x_3 - \alpha_{03}x_4 = q_{10}, \qquad \alpha_{01}x_1 + \beta_{01}x_2 + \alpha_{03}x_3 + \beta_{03}x_4 = q_{20}, \\
p_1x_1 + q_1x_2 + p_3x_3 + q_3x_4 = u_{10}, \qquad -q_1x_1 + p_1x_2 - q_3x_3 + p_3x_4 = u_{20}.
\end{gathered} \tag{3.19}$$

The determinant of Eqs. (3.19) can be written in convenient form. After some transformations, Δ_1^* is expressed as

$$\begin{gathered}
\Delta_1^* = [(\operatorname{coSh}_{01}(h))^2 + (\operatorname{siCh}_{01}(h))^2][(\operatorname{coSh}_{03}(h))^2 + (\operatorname{siCh}_{03}(h))^2] \\
\times [(P_1^*R_3^*)^2 + (P_3^*R_1^*)^2 - 2(P_1^*P_3^*R_1^*R_3^*)\cos(\psi_1 - \psi_3 + \varphi_3 - \varphi_1)] > 0, \\
P_j^* = \sqrt{p_j^2 + q_j^2}, \qquad \psi_j = \arctan\,(q_j/p_j), \qquad j = 1,3.
\end{gathered} \tag{3.20}$$

From the closed system (3.17), we obtain the constants $F_1,\ldots,F_4$, whose explicit expressions are cumbersome and are not given here.

In accordance with (1.1), the boundary condition for the tangential stress in (1.4) can be written as

$$\tau_n\Big|_\Gamma = 2\mu\gamma_n\Big|_\Gamma + 2\eta\,\frac{\partial}{\partial t}\,\gamma_n\Big|_\Gamma = \tau_{10}\cos\omega t + \tau_{20}\sin\omega t. \tag{3.21}$$

If the normal direction on the boundary Γ is defined by the unit vector $\boldsymbol{n} = (n_x, n_y)$, then the tangential direction on Γ for the plane problem is defined by the unit vector $\boldsymbol{\tau} = (-n_y, n_x)$. Then, the tangential component of the displacement vector on Γ is expressed as

$$u_\tau\Big|_\Gamma = (-un_y + vn_x)\Big|_\Gamma.$$

Since in boundary conditions (1.4) on Γ, the normal component u_n is specified to be constant at the points of the boundary, the expressions for the shear γ_n and the shear rate $\partial\gamma_n/\partial t$ can be simplified:

$$2\gamma_n\Big|_\Gamma = \frac{\partial u_\tau}{\partial n}\Big|_\Gamma = \Big(\frac{\partial v}{\partial n}\, n_x - \frac{\partial u}{\partial n}\, n_y\Big)\Big|_\Gamma, \qquad 2\,\frac{\partial}{\partial t}\,\gamma_n\Big|_\Gamma = \frac{\partial}{\partial t}\Big(\frac{\partial v}{\partial n}\, n_x - \frac{\partial u}{\partial n}\, n_y\Big)\Big|_\Gamma.$$

As a result, boundary conditions (3.21) on the triangle side $\xi_3 = 0$ have the form

$$\begin{aligned} &\mu\,\frac{\partial}{\partial n_3}\,(V_1 n_{3x} - U_1 n_{3y})\Big|_{\xi_3=0} + \eta\omega\,\frac{\partial}{\partial n_3}\,(V_2 n_{3x} - U_2 n_{3y})\Big|_{\xi_3=0} = \tau_{10}, \\ &\mu\,\frac{\partial}{\partial n_3}\,(V_2 n_{3x} - U_2 n_{3y})\Big|_{\xi_3=0} - \eta\omega\,\frac{\partial}{\partial n_3}\,(V_1 n_{3x} - U_1 n_{3y})\Big|_{\xi_3=0} = \tau_{20}. \end{aligned} \tag{3.22}$$

Equation (3.22) implies

$$\frac{\partial}{\partial n_3}\,(V_j n_{3x} - U_j n_{3y})\Big|_{\xi_3=0} = \tau_j^*, \qquad \tau_j^* = \frac{\mu\tau_{j0} + (-1)^j\eta\omega\tau_{(3-j)0}}{\mu^2 + \eta^2\omega^2} \qquad (j = 1, 2). \tag{3.23}$$

Substitution of U_j and V_j from (3.11) into the left part of boundary conditions (3.23) yields two equations

$$\begin{aligned} &\frac{\partial}{\partial n_3}\,[-P_j^{(a)}(\xi_1)\boldsymbol{n}_1 \times \boldsymbol{n}_3 - P_j^{(a)}(\xi_2)\boldsymbol{n}_2 \times \boldsymbol{n}_3 \\ &+ Q_j^{(s)}(\xi_1)\boldsymbol{n}_1\boldsymbol{n}_3 + Q_j^{(s)}(\xi_2)\boldsymbol{n}_2\boldsymbol{n}_3 + Q_j^{(s)}(\xi_3)]\Big|_{\xi_3=0} = \tau_j^* \qquad (j = 1, 2). \end{aligned} \tag{3.24}$$

Equation (3.24) can be simplified using the following property for the derivatives:

$$\frac{\partial}{\partial n_3}\,F(\xi_j) = F'(\xi_j)\boldsymbol{n}_j\boldsymbol{n}_3 = -\frac{1}{2}\,F'(\xi_j) \quad (j = 1, 2), \qquad \frac{\partial}{\partial n_3}\,F(\xi_3) = F'(\xi_3). \tag{3.25}$$

By virtue of (3.25) and properties (3.2), boundary conditions (3.24) become

$$\begin{aligned} &(\sqrt{3}/4)[P_j^{(a)\prime}(2h - \xi_1) - P_j^{(a)\prime}(\xi_1)] \\ &+ (1/4)[Q_j^{(s)\prime}(\xi_1) + Q_j^{(s)\prime}(2h - \xi_1)] + Q_j^{(s)\prime}(0) = \tau_j^* \qquad (j = 1, 2). \end{aligned} \tag{3.26}$$

By virtue of properties (3.9) and (3.26), the expressions in square brackets vanish, and, hence,

$$Q_j^{(s)\prime}(0) = \tau_j^* \qquad (j = 1, 2).$$

From this, we find the coefficients G_1 and G_2:

$$\begin{aligned} G_1 &= [\tau_1^*(\beta_{00}\,\mathrm{siCh}_{00}(h) - \alpha_{00}\,\mathrm{coSh}_{00}(h)) + \tau_2^*(\alpha_{00}\,\mathrm{siCh}_{00}(h) + \beta_{00}\,\mathrm{coSh}_{00}(h))]/\Delta_{q1}, \\ G_2 &= [\tau_2^*(\beta_{00}\,\mathrm{siCh}_{00}(h) - \alpha_{00}\,\mathrm{coSh}_{00}(h)) - \tau_1^*(\alpha_{00}\,\mathrm{siCh}_{00}(h) + \beta_{00}\,\mathrm{coSh}_{00}(h))]/\Delta_{q1}. \end{aligned} \tag{3.27}$$

The determinant Δ_{q1} is expressed as

$$\Delta_{q1} = (\alpha_{00}^2 + \beta_{00}^2)[\cosh\,(\alpha_{00}2h) - \cos\,(\beta_{00}2h)]/2 > 0. \tag{3.28}$$

From inequality (3.28), it follows that solution (3.27) is unique. All expressions of the first exact solution of problem (1.2)–(1.4) for a viscoelastic rod of triangular cross section are cumbersome; therefore, we will not give its final form and only indicate the sequence of calculations that lead to this solution: the displacements u and v and the temperature T are determined from (1.6), the amplitudes U_j, V_j, and T_j from (3.11), $P_j^{(a)}$, $R_j^{(s)}$, and $Q_j^{(s)}$ from (3.10), the coefficients $F_1, \ldots, F_4$ from the algebraic system (3.17), $G_1$and G_2 from (3.20), and the determinants Δ_1^* and Δ_{q1} from (3.20) and (3.28). In numerical implementation of the solution, all manipulations should be performed in the reverse order: the determinants Δ_1^* and Δ_{q1} are first calculated from (3.20) and (3.28), the coefficients $F_1, \dots, F_4$ are then found from (3.17), G_1 and G_2 from (3.27), etc. The displacements u and v and the temperature T are expressed in terms of continuous and differentiable functions; therefore, the temperature, strains and strain rates can be found from the well-known formulas of the linear theory of thermoviscoelasticity and the stress is found from (1.1).

4. Second Exact Solution. The solution of problem (1.7)–(1.9) with boundary conditions (1.5) can be represented as the sums

$$U_j(x,y) = \sum_{k=1}^{3}[P_j^{(s)}(\xi_k)n_{kx} - Q_j^{(a)}(\xi_k)n_{ky}], \qquad T_j(x,y) = \sum_{k=1}^{3} R_j^{(a)}(\xi_k),$$
$$V_j(x,y) = \sum_{k=1}^{3}[P_j^{(s)}(\xi_k)n_{ky} + Q_j^{(a)}(\xi_k)n_{kx}], \qquad j = 1,2, \tag{4.1}$$

where

$$P_1^{(s)}(\xi) = \sum_{k=1}^{2}\{p_{2k-1}[F_{2k}^* \operatorname{coCh}_{0(2k-1)}(\xi) - F_{2k-1}^* \operatorname{siSh}_{0(2k-1)}(\xi)]$$
$$- q_{2k-1}[F_{2k-1}^* \operatorname{coCh}_{0(2k-1)}(\xi) + F_{2k}^* \operatorname{siSh}_{0(2k-1)}(\xi)]\},$$

$$P_2^{(s)}(\xi) = \sum_{k=1}^{2}\{q_{2k-1}[F_{2k-1}^* \operatorname{siSh}_{0(2k-1)}(\xi) - F_{2k}^* \operatorname{coCh}_{0(2k-1)}(\xi)]$$
$$- p_{2k-1}[F_{2k-1}^* \operatorname{coCh}_{0(2k-1)}(\xi) + F_{2k}^* \operatorname{siSh}_{0(2k-1)}(\xi)]\},$$

$$Q_1^{(a)}(\xi) = G_1^* \operatorname{coSh}_{00}(\xi) + G_2^* \operatorname{siCh}_{00}(\xi), \qquad Q_2^{(a)}(\xi) = G_2^* \operatorname{coSh}_{00}(\xi) - G_1^* \operatorname{siCh}_{00}(\xi),$$

$$R_1^{(a)}(\xi) = F_1^* \operatorname{coSh}_{01}(\xi) + F_2^* \operatorname{siCh}_{01}(\xi) + F_3^* \operatorname{coSh}_{03}(\xi) + F_4^* \operatorname{siCh}_{03}(\xi),$$

$$R_2^{(a)}(\xi) = F_2^* \operatorname{coSh}_{01}(\xi) - F_1^* \operatorname{siCh}_{01}(\xi) + F_4^* \operatorname{coSh}_{03}(\xi) - F_3^* \operatorname{siCh}_{03}(\xi),$$

$$F_1^* = 2(A_{01} - A_{02}), \qquad F_2^* = 2(C_{01} - C_{02}), \qquad F_3^* = 2(A_{03} - A_{04}),$$

$$F_4^* = 2(C_{03} - C_{04}), \qquad G_1^* = 2(D_1 - D_3), \qquad G_2^* = 2(D_2 + D_4).$$

In the construction of the second exact solution with the boundary conditions having the form (1.5), the condition for the tangential component of the displacement vector $u_\tau\Big|_\Gamma = (vn_x - un_y)\Big|_\Gamma$ implies

$$(V_jn_x - U_jn_y)\Big|_\Gamma = v_{j0}, \qquad j = 1,2. \tag{4.2}$$

Substitution of (4.1) into (4.2) for $\xi_3 = 0$ yields the expression

$$-[P_j^{(s)}(\xi_1)(\boldsymbol{n}_1 \times \boldsymbol{n}_3) + P_j^{(s)}(2h - \xi_1)(\boldsymbol{n}_2 \times \boldsymbol{n}_3)]$$
$$+ [Q_j^{(a)}(\xi_1)(\boldsymbol{n}_1\boldsymbol{n}_3) + Q_j^{(a)}(2h - \xi_1)(\boldsymbol{n}_2\boldsymbol{n}_3)] + Q_j^{(a)}(0) = v_{j0} \qquad (j = 1,2). \tag{4.3}$$

Using the properties (3.2) and (3.9), one can show that the expressions in square brackets containing the variable ξ_1 vanish; therefore, Eq. (4.3) implies

$$Q_j^{(a)}(0) = v_{j0} \qquad (j = 1,2). \tag{4.4}$$

Let us write two equations (4.4) for G_1^* and G_2^*:

$$G_1^* \cos(\beta_{00}h) \sinh\,(\alpha_{00}h) + G_2^* \sin(\beta_{00}h) \cosh\,(\alpha_{00}h) = -v_{10},$$

$$-G_1^* \sin(\beta_{00}h) \cosh\,(\alpha_{00}h) + G_2^* \cos(\beta_{00}h) \sinh\,(\alpha_{00}h) = -v_{20}.$$

The determinant of these equations $\Delta_{q2} > 0$; therefore, this system has the solution

$$G_1^* = [v_{20} \sin(\beta_{00}h) \cosh\,(\alpha_{00}h) - v_{10} \cos\,(\beta_{00}h) \sinh\,(\alpha_{00}h)]/\Delta_{q2},$$

$$G_2^* = -[v_{10}\sin(\beta_{00}h)\cosh\,(\alpha_{00}h) + v_{20}\cos\,(\beta_{00}h)\sinh\,(\alpha_{00}h)]/\Delta_{q2},$$

$$\Delta_{q2} = \cosh\,(2\alpha_{00}h) - \cos\,(2\beta_{00}h) > 0.$$

To satisfy the second boundary condition in (1.5), we transform the expression for the normal stress. Since on the boundary Γ, the tangential component u_τ is constant, the stress $\sigma_n\Big|_\Gamma$ can be represented as

$$\sigma_n\Big|_\Gamma = \lambda_0\,\frac{\partial u_n}{\partial n}\Big|_\Gamma + \zeta_0\,\frac{\partial}{\partial n}\,\frac{\partial u_n}{\partial t}\Big|_\Gamma = \sigma_{10}\cos\omega t + \sigma_{20}\sin\omega t. \tag{4.5}$$

Substitution of $u_n\Big|_\Gamma$ from expression (3.12) into (4.5) yields

$$\begin{aligned} &\lambda_0\,\frac{\partial}{\partial n_3}\,(U_1 n_{3x} + V_1 n_{3y})\Big|_{\xi_3=0} + \zeta_0\omega\,\frac{\partial}{\partial n_3}\,(U_2 n_{3x} + V_2 n_{3y})\Big|_{\xi_3=0} = \sigma_{10},\\ &\lambda_0\,\frac{\partial}{\partial n_3}\,(U_2 n_{3x} + V_2 n_{3y})\Big|_{\xi_3=0} - \zeta_0\omega\,\frac{\partial}{\partial n_3}\,(U_1 n_{3x} + V_1 n_{3y})\Big|_{\xi_3=0} = \sigma_{20}. \end{aligned} \tag{4.6}$$

Equation (4.6) implies

$$\frac{\partial}{\partial n_3}\,(U_j n_{3x} + V_j n_{3y})\Big|_{\xi_3=0} = N_j, \qquad N_j = \frac{\lambda_0\sigma_{j0} + (-1)^j\zeta_0\omega\sigma_{(3-j)\,0}}{\lambda_0^2 + \zeta_0^2\omega^2} \qquad (j = 1, 2). \tag{4.7}$$

Substitution of U_j and V_j from (4.1) into the left side of boundary conditions (4.7) yields two equations

$$\begin{aligned} &\frac{\partial}{\partial n_3}\,[P_j^{(s)}(\xi_1)\boldsymbol{n}_1\boldsymbol{n}_3 + P_j^{(s)}(\xi_2)\boldsymbol{n}_2\boldsymbol{n}_3 + P_j^{(s)}(\xi_3)\\ &+ Q_j^{(a)}(\xi_1)\boldsymbol{n}_1\times\boldsymbol{n}_3 + Q_j^{(a)}(\xi_2)\boldsymbol{n}_2\times\boldsymbol{n}_3]\Big|_{\xi_3=0} = N_j \qquad (j = 1, 2). \end{aligned} \tag{4.8}$$

In view of the properties (3.2) and (3.23), boundary conditions (4.8) become

$$\begin{aligned} &(1/4)[P_j^{(s)\prime}(\xi_1) + P_j^{(s)\prime}(h - \xi_1)]\\ &+ (\sqrt{3}/4)[Q_j^{(a)\prime}(h - \xi_1) - Q_j^{(a)\prime}(\xi_1)] + P_j^{(s)\prime}(0) = N_j \qquad (j = 1, 2). \end{aligned} \tag{4.9}$$

From (3.9) it follows that expressions in square brackets vanish; therefore, from (4.9) we obtain two equations

$$P_j^{(s)\prime}(0) = N_j \qquad (j = 1, 2). \tag{4.10}$$

It remains to satisfy the boundary condition in (1.5) for the temperature on the triangle side $\xi_3 = 0$. After substitution of $T_j(x, y)$ from (4.1), this condition becomes

$$[R_j^{(a)}(\xi_1) + R_j^{(a)}(2h - \xi_1)] + R_j^{(a)}(0) = T_{j0}, \qquad j = 1, 2. \tag{4.11}$$

Using (3.9), it is easy to show that the expression in square brackets vanish. Then, from (4.11) we obtain

$$R_j^{(a)}(0) = T_{j0}, \qquad j = 1, 2. \tag{4.12}$$

The system of four equations (4.10), (4.12) for the coefficients $F_1^*, \ldots, F_4^*$ is written in explicit form

$$\begin{aligned} \sum_{k=1}^{2}\{&q_{2k-1}F_{2k-1}^*[\alpha_{0(2k-1)}\,\mathrm{coSh}_{0(2k-1)}(h) - \beta_{0(2k-1)}\,\mathrm{siCh}_{0(2k-1)}(h)]\\ &+ q_{2k-1}F_{2k}^*[\alpha_{0(2k-1)}\,\mathrm{siCh}_{0(2k-1)}(h) + \beta_{0(2k-1)}\,\mathrm{coSh}_{0(2k-1)}(h)]\\ &+ p_{2k-1}F_{2k}^*[\beta_{0(2k-1)}\,\mathrm{siCh}_{0(2k-1)}(h) - \alpha_{0(2k-1)}\,\mathrm{coSh}_{0(2k-1)}(h)]\\ &+ p_{2k-1}F_{2k-1}^*[\alpha_{0(2k-1)}\,\mathrm{siCh}_{0(2k-1)}(h) + \beta_{0(2k-1)}\,\mathrm{coSh}_{0(2k-1)}(h)]\} = N_1, \end{aligned}$$

$$\sum_{k=1}^{2}\{q_{2k-1}F_{2k}^*[\alpha_{0(2k-1)}\,\mathrm{coSh}_{0(2k-1)}(h) - \beta_{0(2k-1)}\,\mathrm{siCh}_{0(2k-1)}(h)]$$

$$
\begin{aligned}
&- q_{2k-1}F^*_{2k-1}[\alpha_{0(2k-1)}\,\mathrm{siCh}_{0(2k-1)}(h) + \beta_{0(2k-1)}\,\mathrm{coSh}_{0(2k-1)}(h)]\\
&+ p_{2k-1}F^*_{2k-1}[\alpha_{0(2k-1)}\,\mathrm{coSh}_{0(2k-1)}(h) - \beta_{0(2k-1)}\,\mathrm{siCh}_{0(2k-1)}(h)]\\
&+ p_{2k-1}F^*_{2k}[\alpha_{0(2k-1)}\,\mathrm{siCh}_{0(2k-1)}(h) + \beta_{0(2k-1)}\,\mathrm{coSh}_{0(2k-1)}(h)]\} = N_2,\\
&\sum_{k=1}^{2}[F^*_{2k-1}\,\mathrm{coSh}_{0(2k-1)}(h) + F^*_{2k}\,\mathrm{siCh}_{0(2k-1)}(h)] = -T_{10},\\
&\sum_{k=1}^{2}[F^*_{2k}\,\mathrm{coSh}_{0(2k-1)}(h) - F^*_{2k-1}\,\mathrm{siCh}_{0(2k-1)}(h)] = -T_{20}.
\end{aligned}
\tag{4.13}
$$

The linear system of four equations (4.13) is easily solved on a computer. It remains to elucidate whether there are cases where these equations have no solution. We prove that the determinant of the system is always $\Delta^*_2 > 0$. To write the determinant in convenient form, we use the properties of Eqs. (4.13) and replace the coefficients $F^*_1, \ldots, F^*_4$ by new unknown complexes $x^*_1, \ldots, x^*_4$:

$$
\begin{aligned}
x^*_1 &= F^*_1\,\mathrm{siCh}_{01}(h) - F^*_2\,\mathrm{coSh}_{01}(h), \qquad x^*_2 = F^*_1\,\mathrm{coSh}_{01}(h) + F^*_2\,\mathrm{siCh}_{01}(h),\\
x^*_3 &= F^*_3\,\mathrm{siCh}_{03}(h) - F^*_4\,\mathrm{coSh}_{03}(h), \qquad x^*_4 = F^*_3\,\mathrm{coSh}_{03}(h) + F^*_4\,\mathrm{siCh}_{03}(h).
\end{aligned}
\tag{4.14}
$$

In notation (4.14), system (4.13) is simplified:

$$
\begin{aligned}
&(p_1\alpha_{01} - q_1\beta_{01})x_1 + (p_1\beta_{01} + q_1\alpha_{01})x_2 + (p_3\alpha_{03} - q_3\beta_{03})x_3 + (p_3\beta_{03} + q_3\alpha_{03})x_4 = N_1,\\
&(p_1\alpha_{01} - q_1\beta_{01})x_2 - (p_1\beta_{01} + q_1\alpha_{01})x_1 - (p_3\beta_{03} + q_3\alpha_{03})x_3 + (p_3\alpha_{03} - q_3\beta_{03})x_4 = N_2,\\
&-x_1 - x_3 = -T_{20}, \qquad x_2 + x_4 = -T_{10}.
\end{aligned}
\tag{4.15}
$$

The determinant of Eqs. (4.15) can be written in compact form. After some transformations for Δ^*_2, we obtain the expression

$$
\begin{aligned}
\Delta^*_2 &= [(\mathrm{coSh}_{01}(h))^2 + (\mathrm{siCh}_{01}(h))^2][(\mathrm{coSh}_{03}(h))^2 + (\mathrm{siCh}_{03}(h))^2]\\
&\times [(q_1\beta_{01} - p_1\alpha_{01} - q_3\beta_{03} + p_3\alpha_{03})^2 + (p_1\beta_{01} + q_1\alpha_{01} - p_3\beta_{03} - q_3\alpha_{03})^2] > 0.
\end{aligned}
$$

From the closed system of equations (4.13), we find the constants $F^*_1, \ldots, F^*_4$, whose explicit expressions are cumbersome and are not given here.

In a thermoviscoelastic rod, the propagation of one temperature and two elastic waves (shear and longitudinal) is possible. The characteristics of these waves are determined by the real and imaginary parts of the roots α_j and β_j $(j = 1, \ldots, 4)$. To establish which roots correspond to the waves listed above, we set the coupling and viscosity coefficients in Eq. (2.13) equal to zero ($k = \zeta_0 = 0$), i.e., $M_e = M_v = M_\zeta = 0$. Then, from (2.14) we obtain

$$
\alpha_{1,2} = \pm\sqrt{\frac{\omega}{2b}}\,(1+i), \qquad \alpha_{3,4} = \pm\sqrt{N_0\,\frac{\omega}{b}} = \pm\omega\sqrt{\frac{\rho}{\lambda_0}}\,.
$$

From this it follows that the roots $\alpha_{1,2}$ define the parameters of the temperature wave and the roots $\alpha_{3,4}$ define the parameters of the longitudinal elastic wave. Because the model is coupled, both the temperature and elastic strains change in the temperature wave, and the temperature also changes in the longitudinal elastic wave. Only the shear wave does not influence the temperature field. Generally, the velocities of the temperature (v_T), shear (v_μ), and longitudinal (v_λ) elastic waves can be calculated by the formulas

$$
v_T = \omega/\beta_{01}, \qquad v_\mu = \omega/\beta_{00}, \qquad v_\lambda = \omega/\beta_{03}. \tag{4.16}
$$

The lengths of these waves are determined from the expressions

$$
L_T = 2\pi/\beta_{01}, \qquad L_\mu = 2\pi/\beta_{00}, \qquad L_\lambda = 2\pi/\beta_{03}. \tag{4.17}
$$

Formulas (4.16) and (4.17) and experimental data can be used to calculate the rheological characteristics of thermoviscoelastic materials. For example, the viscosity coefficients of many solids have not yet been determined. From the formulas for the characteristic roots, it follows that the temperature and strain fields are significantly affected by the dimensionless parameter R_0. In addition, a decrease in the coupling coefficient k leads to a decrease in the parameters M_e, M_v, and R_0. Thus, if the parameter R_0 is small, the coupling in the formulation of the problem can be ignored and if $R_0 \sim 1$ or $R_0 > 1$, the coupling should be considered. Account of the coupling also depends on the required calculation accuracy in the solution of the problem.

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